Pseudoclassical Mechanics for the spin 0 and 1 particles.

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Abstract

We give an action for the massless spinning particle in pseudoclassical mechanics by using grassmann variables. The constructed action is invariant under τ-reparametrizations, local SUSY and $O(N)$ transformations. After quantization, for the special case $N = 2$, we get an action which describes the spin 0, 1 particles and topological sectors of the massless DKP theory. A SUSY formulation of the given model is also explored.

1 Introduction

Investigations of particle systems with arbitrary spin was initially given by Bargmann-Wigner [1] and Rarita-Schwinger [2], here the Dirac representations of the spin one half particles are the basis to the construction of higher spin theories. The formalism is based on the bispinor wave function with $2s$ Dirac indices (for spin s) and the total symmetrical representation is used to study the maximum spin value of the model.

On the other hand, the first ideas about the studies of classical systems that include in the phase space both commuting and anticommuting variables (pseudoclassical mechanics) was put forward by Schwinger [3] in 1953. However it was Martin [4] who achieved these ideas in 1959. Later in the Berezin and Marinov works [5] a model for the description of spin one half particles was proposed, here the consistent formulation of the relativistic particle dynamics implies in the addition of a new constraint, this is because the formulation of the massive case has five grassmann variables. At the same time these models were also studied by Casalbuoni [6] who explored the internal group symmetry and the gauge invariance of the resulting action. In this way was possible the description of spinless and spin one particles using these internal symmetries. Interaction of spinning particle systems with external Yang-Mills and gravitational fields was investigated in [7]. The quantization of similar models are performed by means of the Dirac procedure for constrained systems.

Many other papers appeared about the study of spinning particles in the framework of pseudoclassical mechanics, for example the derivation of the equation of motions for the massive and massless spinning particles are treated in the works [8, 9, 10, 11], where the spin description is achieved by means of the inclusion of internal group symmetries. Similarly, the case of the Dirac particle is discussed in the works [12, 13, 14]. A path integral representation for obtaining a Dirac propagator was also obtained in [15] and other studies connecting the pseudoclassical mechanics with the string theory was investigated [16] for the free case as in interacting with an external field. Also, the pseudoclassical description of massless Weyl fermions and its path integral quantization when coupled to Yang-Mills and gravitational fields was studied in [17]. Similarly, the path integral quantization of spinning particles interacting with external electromagnetic field was analyzed in [18].

Besides this, the pseudoclassical approach can be applied to other different models. This is the case of the Duffin-Kemmer-Petiau (DKP) theory [19, 20, 21] which describes massive spin 0 and spin 1 particles in a unified representation. Questions about the equivalence of the DKP theory with theories like Klein-Gordon and Maxwell are discussed in [22, 23, 24] (a good historical review of the DKP theory can be found in [25, 26]). The Field theory of the massless DKP has a local gauge symmetry which describes the electromagnetic field in its spin 1 sector. It is important to notice that the massless case can not be obtained through the limit $m \to 0$ of the massive case. This is due to the fact that the projections of DKP field into spin 0 and 1 sectors involve the mass as a multiplicative factor [30] so that taking the limit $m \to 0$ makes the results previously obtained useless. Moreover,

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if we simply make mass equal to zero in the usual massive DKP Lagrangian we obtain a Lagrangian with no local gauge symmetry. Studies in the Riemann-Cartan space time was proposed in [28, 29, 30].

Recently, a super generalization of the DKP algebra was done by Okubo [31] where the starting point is the study of all irreducible representations by means of the Lie algebra $so(1,4)$ [32], moreover, a paraDKP (PDKP) algebra is constructed intimately related to the Lie superalgebra $osp(1,4)$, obtaining as result the super DKP algebra that contains the boson and fermion representations.

An extended variant including Grassmann variables for the DKP theory is very interesting for many reasons, for example a pseudoclassical version allow us to make an attempt to the construction of a supersymmetry variant of the theory where the action must be expressed in terms of (super)fields. It is also no clear about the particle states that will compose the (super)multiplet in this theory.

In this work we propose a possible action for the massless DKP theory in the pseudoclassical approach. In section **2**, the pseudoclassical action is given including the correct boundary terms that yields a consistent equations of motions. We carry out the constraint analysis of the system and verify his invariance under τ-reparametrizations, internal group $O(N)$ and SUSY transformations. We find the generators of corresponding transformations and give the Pauli-Lubanski vector. In section **3**, the quantization is performed and proved that for the special case $N=2$ the both sectors of spin 0 and spin 1 of the DKP theory appear. We get the scalar and vectorial field as a first result, we also obtain the topological field solutions correspondent to the both spin sectors. In Section **4**, using the SUSY principles we extend the proposed action to the Superspace formalism obtaining a consistent result as in the pseudoclassical model. Finally in section **5**, we give our conclusions and comments.

2 Pseudoclassical Mechanics

We start with the action in the first order formalism that considers an internal group symmetry

$$S=\int_{\tau_1}^{\tau_2} d\tau\left[\left(\dot{x}-i\chi\psi\right)p+\frac{e}{2}p^2+\frac{i}{2}\psi\dot{\psi}+\frac{i}{2}f\psi\psi\right]+\frac{i}{2}\psi\left(\tau_2\right)\psi\left(\tau_1\right) \tag{1}$$

here x_μ is the space time coordinate, p_μ the auxiliary momentum vector; $\psi_\mu^k\left(\tau\right)-k,l,...=1,2,...N$ are the fermion coordinates, superpartner of $x_\mu\left(\tau\right)$, $\left(x_\mu,\psi_\mu^k\right)$ is the multiplet of matter; $e\left(\tau\right)$ is the *einbein*, his superpartner $\chi_k\left(\tau\right)$ is the unidimensional gravitino; $f_{ik}\left(\tau\right)=-f_{ki}\left(\tau\right)$ is the gauge field for internal symmetry, $\left(e,\chi_k,f_{ik}\right)$ is the supergravitational multiplet on the world line.

The action (1) includes the correct boundary terms that guarantee the consistence of the equations of motions for the grassmann variables. This is because in the variational principle the fermionic canonical coordinates have only one condition

$$\delta\left(\psi\left(\tau_2\right)+\psi\left(\tau_1\right)\right)=0 \tag{2}$$

for the other coordinates only the space time coordinate is restricted to the condition

$$\delta x\left(\tau_2\right)=\delta x\left(\tau_1\right)=0 \tag{3}$$

internal group indices in the case $N=2$ when $i,k=1,2$ are contracted by means of symbol Kroeneker δ_{ik} (for the group $O\left(2\right)$ and spin 1) or Levi-Civita symbol ϵ_{ik} (for the group $Sp\left(1\right)$ and spin 0).

The lagrangian that follows from (1) is

$$\mathcal{L}=\left(\dot{x}-i\chi\psi\right)p+\frac{e}{2}p^2+\frac{i}{2}\psi\dot{\psi}+\frac{i}{2}f\psi\psi \tag{4}$$

It is possible to write the action (1) in a different way, for this we perform the variation of S with respect to p, then we get the following equation

$$p=-e^{-1}\left(\dot{x}-i\chi\psi\right) \tag{5}$$

inserting this solution into (1) we obtain the second order formalism of the action

$$\begin{aligned} S &= \int_{\tau_1}^{\tau_2} d\tau\left[-\frac{e^{-1}}{2}\left(\dot{x}^2-2i\dot{x}\chi\psi-\left(\chi\psi\right)^2\right)+\frac{i}{2}\psi\dot{\psi}+\frac{i}{2}f\psi\psi\right] \\ &\quad +\frac{i}{2}\psi\left(\tau_2\right)\psi\left(\tau_1\right) \end{aligned} \tag{6}$$

then the lagrangian that follows from (6) is

$$\mathcal{L} = -\frac{e^{-1}}{2}\left(\dot{x}^2 - 2i\dot{x}\chi\psi - (\chi\psi)^2\right) + \frac{i}{2}\psi\dot{\psi} + \frac{i}{2}f\psi\psi \tag{7}$$

the term $(\chi\psi)^2 = \chi_i\psi_i\chi_k\psi_k$ appears because an internal group symmetry $O(N)$ was introduced in the theory.

Both formulations (1) and (6) are equivalent and as we will see later the constraint analysis gives the same result.

Equations of motions that follow from the action (1) result in

$$p_\mu\psi_k^\mu = 0, \quad \psi_{\mu i}\psi_k^\mu = 0, \quad \dot{\psi}_k^\mu = -p^\mu\chi_k + f_{ik}\psi_i^\mu, \quad \dot{p} = 0$$

we can see that for a special case $e = 1$, $\chi = f = 0$ we obtain the solutions

$$x_\mu(\tau) = x_\mu(0) + p_\mu\tau, \quad \psi_k^\mu = const.$$

2.1 Constraint Analysis

Now we proceed to the constraint analysis of the theory. Using the definition for the canonical momentum: $p_a = \frac{\overleftarrow{\partial}\mathcal{L}}{\partial\dot{q}^a}$, we obtain

$$\begin{aligned} p_\mu &= \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = p_\mu; \quad \pi_\mu^k = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_k^\mu} = \frac{i}{2}\psi_\mu^k \\ \pi &= \frac{\partial\mathcal{L}}{\partial\dot{e}^\mu} = 0; \quad \pi^k = \frac{\partial\mathcal{L}}{\partial\dot{\chi}_k} = 0; \quad \pi^{ik} = \frac{\partial\mathcal{L}}{\partial\dot{f}_{ik}} = 0 \end{aligned} \tag{8}$$

from which a set of primary constraints appears

$$\Omega_\mu^k = \pi_\mu^k - \frac{i}{2}\psi_\mu^k \approx 0, \quad \Omega_\pi = \pi \approx 0, \quad \Omega^k = \pi^k \approx 0, \quad \Omega^{ik} = \pi^{ik} \approx 0 \tag{9}$$

following the standard Dirac procedure for a theory with constraints we construct the primary hamiltonian from the lagrangian (4), $\mathcal{H} = p_a\dot{q}^a - \mathcal{L}$,

$$\mathcal{H}^{(1)} = i\chi_k\psi_k^\mu p_\mu - \frac{e}{2}p^2 - \frac{i}{2}f_{ik}\psi_{\mu i}\psi_k^\mu + \lambda^a\Omega_a \tag{10}$$

where we have included the primary constraints (9), $\lambda^a = \left\{\lambda_\mu^k, \lambda_\pi, \lambda^k, \lambda^{ik}\right\}$ are the lagrange multipliers. When we apply the stability conditions on the primary constraints

$$\dot{\Omega}_a = \left\{\Omega_a, \mathcal{H}^{(1)}\right\}_{PB} = 0 \tag{11}$$

we obtain a new set of secondary constraints

$$\Omega_\pi^{(2)} = \frac{1}{2}p^2 \approx 0, \quad \Omega_k^{(2)} = i\psi_k^\mu p_\mu \approx 0, \quad \Omega_{ik}^{(2)} = i\psi_{\mu i}\psi_k^\mu \approx 0 \tag{12}$$

the conservation of these secondary constraints in time tell us that no more constraints appear in the theory. Next the constraint classification gives the following first class

$$\Omega_\pi^{(2)} = \frac{1}{2}p^2 \approx 0 \tag{13}$$

$$\Omega_k^{(2)} = i\psi_k^\mu p_\mu \approx 0 \tag{14}$$

$$\Omega_{ik}^{(2)} = i\psi_i^\mu\psi_k^\mu \approx 0 \tag{15}$$

and the second class constraints

$$\Omega_\mu^k = \pi_\mu^k - \frac{i}{2}\psi_\mu^k \approx 0 \tag{16}$$

with the help of the second class constraints we construct the Dirac Bracket (DB) between the canonical variables and obtain

$$\left\{\psi_\mu^i, \psi_\nu^k\right\}_{DB} = -i\delta^{ik}g_{\mu\nu}, \quad \{x_\mu, p_\nu\}_{DB} = g_{\mu\nu} \tag{17}$$

2.2 Invariance

In the theory with the action (6), we have three gauge transformations that do not change their physical sense. The τ-reparametrization

$$\begin{aligned} \delta x &= \varepsilon \dot{x}, \quad \delta \psi = \varepsilon \dot{\psi} \\ \delta e &= (\varepsilon e)^{\cdot}, \quad \delta \chi = (\varepsilon \chi)^{\cdot}, \quad \delta f = (\varepsilon f)^{\cdot} \end{aligned} \tag{18}$$

the invariance under local internal symmetries $O(N)$

$$\begin{aligned} \delta x &= 0, \quad \delta \psi = a\psi \\ \delta e &= 0, \quad \delta \chi = a\chi, \quad \delta f = \dot{a} + af - fa \end{aligned} \tag{19}$$

and the invariance under local $(n = 1)$ SUSY transformations

$$\begin{aligned} \delta x &= i\alpha\psi, \quad \delta \psi = e^{-1}\alpha\left(\dot{x} - i\chi\psi\right) \\ \delta e &= 2i\alpha\chi, \quad \delta \chi = \dot{\alpha} - f\alpha, \quad \delta f = 0 \end{aligned} \tag{20}$$

It is interesting to commute two local $(n = 1)$ SUSY transformations. This gives

$$[\delta_\alpha, \delta_\beta]\, x = \delta_{\varepsilon_0}\dot{x} + \delta_{a_0}x + \delta_{\alpha_0}x \tag{21}$$

$$[\delta_\alpha, \delta_\beta]\, \psi = \delta_{\varepsilon_0}\dot{\psi} + \delta_{a_0}\psi + \delta_{\alpha_0}\psi \tag{22}$$

$$[\delta_\alpha, \delta_\beta]\, e = \delta_{\varepsilon_0}\dot{e} + \delta_{a_0}e + \delta_{\alpha_0}e \tag{23}$$

$$[\delta_\alpha, \delta_\beta]\, \chi = \delta_{\varepsilon_0}\dot{\chi} + \delta_{a_0}\chi + \delta_{\alpha_0}\chi \tag{24}$$

$$[\delta_\alpha, \delta_\beta]\, f = \delta_{\varepsilon_0}\dot{f} + \delta_{a_0}f + \delta_{\alpha_0}f \tag{25}$$

where the new parameters are now field dependent

$$\varepsilon_0 = 2ie^{-1}\alpha\beta, \quad \alpha_0 = -\varepsilon_0\chi, \quad a_0 = -\varepsilon_0 f \tag{26}$$

this shows that there is no simple gauge group structure, although the invariance is still enough to secure good physical properties of the action.

The invariance of the action (6) is reached if we impose the conditions at the endpoints for the parameters

$$\varepsilon(\tau_1) = \varepsilon(\tau_2) = 0, \quad \alpha(\tau_1) = \alpha(\tau_2) = 0 \tag{27}$$

On the other hand it is possible to find the generators of the transformations (18)-(20). We follow the work of Casalbuoni [6] where the generators of the transformations F are given by

$$F = p_a \delta q^a - \varphi, \quad \delta L = \frac{d\varphi}{d\tau} \tag{28}$$

being φ the generating function. To verify the correctness of found generators we use

$$\delta u = \{u, \epsilon F\}_{DB} \tag{29}$$

where ϵ is the parameter of a given transformation.

We find for the τ-reparametrizations

$$F = i\chi\psi p - \frac{e}{2}p^2 - \frac{i}{2}f\psi\psi \tag{30}$$

$$\{x^\mu, \varepsilon F\}_{DB} = \varepsilon\dot{x}^\mu, \quad \{\psi_k^\mu, \varepsilon F\}_{DB} = \varepsilon\dot{\psi}_k^\mu \tag{31}$$

internal $O(N)$ symmetries

$$F_{ik} = \frac{i}{2}\psi_i^\mu\psi_{\mu k} + \chi_i\pi_k \tag{32}$$

$$\{x^\mu, aF\}_{DB} = 0, \quad \{\psi_i^\mu, aF\}_{DB} = a_{ik}\psi_k^\mu, \quad \{\chi_i, aF\}_{DB} = a_{ik}\chi_k \tag{33}$$

and SUSY transformations

$$F_k = ip_\mu\psi_k^\mu + 2i\chi_k\pi \tag{34}$$

$$\{x^\mu, \alpha F\}_{DB} = i\alpha_k\psi_k^\mu, \quad \{\psi, \alpha F\}_{DB} = e^{-1}\alpha\left(\dot{x} - i\chi\psi\right) \tag{35}$$

$$\{e, \alpha F\}_{DB} = 2i\alpha\chi \tag{36}$$

To close the invariance we remark that the proposed theory is also invariant under Poincaré transformations, i.e.

$$\delta x^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu} + \epsilon^{\mu}, \quad \delta\psi^{\mu}_{k} = \omega^{\mu}{}_{\nu}\psi^{\nu}_{k}, \quad \delta e = \delta\chi = \delta f = 0 \tag{37}$$

with the generators

$$\epsilon^{a} F_{a} = \epsilon^{\mu} P_{\mu} + \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} \tag{38}$$

where

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad L_{\mu\nu} = x_{\nu} p_{\mu} - x_{\mu} p_{\nu}, \quad S_{\mu\nu} = i\psi^{k}_{\mu}\psi^{k}_{\nu} \tag{39}$$

in this way is constructed the Pauli-Lubanski vector

$$W_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} P^{\nu} M^{\lambda\rho}, \quad W^{2} = \frac{1}{2}\left(P^{2}S^{2} + 2\left(S_{\mu\nu}P^{\nu}\right)^{2}\right) \tag{40}$$

3 Quantization

The constraint analysis which was done before takes a physical sense when the quantization is performed and a coherent interpretation of the equation of motions is given.

With the quantization the canonical variables becomes operators

$$x_{\mu} \to \widehat{x}_{\mu}, \quad p_{\mu} \to \widehat{p}_{\mu}, \quad \psi^{i}_{\mu} \to \widehat{\psi}^{i}_{\mu} \tag{41}$$

and the DB follows the commutator or anticommutator rules

$$\{\widehat{\ }\} \to i\hbar\{\quad\}_{DB} \tag{42}$$

thus we have the following commutation relations

$$\left\{\widehat{\psi}^{i}_{\mu}, \widehat{\psi}^{k}_{\nu}\right\} = \hbar\delta^{ik} g_{\mu\nu}, \quad \left[\widehat{x}_{\mu}, \widehat{p}_{\mu}\right] = i\hbar g_{\mu\nu} \tag{43}$$

We pick out a general realization for the operator $\widehat{\psi}^{k}_{\mu}$ satisfying the relation (43) and the equations of motions

$$D\left(\widehat{\psi}^{k}_{\mu}\right) = S\left(Y\right)\left(\left(\gamma_{5}\right)^{\otimes(k-1)} \otimes \gamma_{\mu}\gamma_{5} \otimes I^{\otimes(N-k)}\right) \tag{44}$$

here $S(Y)$ is the Young symmetrization operator, γ_{μ} are the Dirac matrices and γ_5 is given by

$$\gamma_{5} = i\gamma_{0}\gamma_{1}\gamma_{2}\gamma_{3}, \quad \left(\gamma_{5}\right)^{2} = 1 \tag{45}$$

The first class constraints are applied into the vector state $|\Phi\rangle \equiv |\Phi\rangle_{\alpha_1\ldots\alpha_N}$. We recall that an internal group symmetry $O(N)$, where $i, k, ... = 1, 2, ..., N$, is considered in the Lagrangian (1). Thus we obtain

$$p^{2}\left|\Phi\right\rangle_{\alpha_1\ldots\alpha_N} = 0 \tag{46}$$

$$p^{\mu}\gamma^{k}_{\mu}\left|\Phi\right\rangle_{\alpha_1\ldots,\alpha_k,\ldots\alpha_N} = 0 \tag{47}$$

$$\gamma^{\mu i}\gamma^{k}_{\mu}\left|\Phi\right\rangle_{\alpha_1\ldots,\alpha_i,\ldots,\alpha_k,\ldots\alpha_N} = 0 \tag{48}$$

the first equation is the mass shell condition in the case of a massless particle. The second one is a set of linear equations for every Dirac indices where no symmetrization on the vector state $|\Phi\rangle$ is assumed. However when the symmetrization over the vector state is taken into account, (47) becomes the Bargmann-Wigner [1] equation for a particle with spin $N/2$. The total symmetrical part of $|\Phi\rangle$ generates a representation with the higher spin value. In our case, the third equation is a projector of the representations of DKP theory, i.e., it separates out a particular spin representation of the vector state.

In the particular choose: $i, k = 1, 2$, i.e. when the internal group symmetry is $O(2)$, (46)-(48) reproduce de DKP equations for massless particles with spin 0 and 1. In this case the realization (44) becomes

$$D\left(\widehat{\psi}^{1}_{\mu}\right) = i\sqrt{\frac{\hbar}{2}}\left(\gamma_{\mu}\gamma_{5} \otimes 1\right), \quad D\left(\widehat{\psi}^{2}_{\mu}\right) = i\sqrt{\frac{\hbar}{2}}\left(\gamma_{5} \otimes \gamma_{\mu}\gamma_{5}\right) \tag{49}$$

Let's take only two Dirac indices in the vector state $|\Phi\rangle_{\alpha_1\alpha_2}$, then using a complete set of Dirac matrices we decompose $|\Phi\rangle_{\alpha_1\alpha_2}$ as follows [33]

$$\begin{aligned}\left|\Phi\right\rangle_{\alpha_1\alpha_2} &= a\left(\gamma^{5}C\right)_{\alpha_1\alpha_2}\zeta_{5} + a_{1}\left(\gamma^{5}\gamma^{\mu}C\right)_{\alpha_1\alpha_2}\zeta_{5\mu} + a_{2}C_{\alpha_1\alpha_2}\zeta \\ &\quad + b\left(\gamma^{\mu}C\right)_{\alpha_1\alpha_2}\left(\zeta_{\mu}\right) + b_{1}\left(\Sigma^{\mu\nu}C\right)_{\alpha_1\alpha_2}\zeta_{\mu\nu}\end{aligned} \tag{50}$$

here a, a_1, b, b_1 and a_2 must be considered as free parameters and will be adjusted to assure the correctness of the final equations. The term: $C_{\alpha_1\alpha_2}\zeta$, is referred to a trivial representation and we do not consider it, therefore, we set $a_2 = 0$.

We also have

$$\Sigma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \tag{51}$$

and C is the charge conjugation matrix

$$C^T = -C. \tag{52}$$

Considering the properties of the matrix C we obtain the antisymmetrical

$$|\Phi\rangle_{[\alpha_1\alpha_2]} = a\ \left(\gamma^5 C\right)_{\alpha_1\alpha_2}\ \zeta_5 + a_1\left(\gamma^5\gamma^\mu C\right)_{\alpha_1\alpha_2}\zeta_{5\mu} \tag{53}$$

and the symmetrical part of the vector state.

$$|\Phi\rangle_{\{\alpha_1\alpha_2\}} = b\left(\gamma^\mu C\right)_{\alpha_1\alpha_2}\zeta_\mu + b_1\left(\Sigma^{\mu\nu} C\right)_{\alpha_1\alpha_2}\zeta_{\mu\nu}. \tag{54}$$

Thus for the particular case of $O(2)$ symmetry we obtain

$$p^2\,|\Phi\rangle_{\alpha_1\alpha_2} = 0 \tag{55}$$

$$p^\mu\gamma_\mu^{(1)}\,|\Phi\rangle_{\alpha_1\alpha_2} = 0, \quad p^\mu\gamma_\mu^{(2)}\,|\Phi\rangle_{\alpha_1\alpha_2} = 0 \tag{56}$$

$$\gamma_\mu^{(1)}\gamma^{\mu(2)}\,|\Phi\rangle_{\alpha_1\alpha_2} = 0 \tag{57}$$

these relations give the DKP equation for spin 0 and spin 1. The relation (57) can be shown to be a projector that separates the corresponding sector of the vector state $|\Phi\rangle_{\alpha_1\alpha_2}$.

3.1 Spin 0

Let's take the antisymmetrical part of the vector state $|\Phi\rangle_{\alpha_1\alpha_2}$ and replace it in one of the equations (56), then we obtain

$$\left(p_\mu\gamma^\mu\right)_{\alpha\alpha_1}|\Phi\rangle_{[\alpha_1\alpha_2]} = \left(p_\mu\gamma^\mu\right)_{\alpha\alpha_1}\left[a\left(\gamma^5 C\right)_{\alpha_1\alpha_2}\zeta_5 + a_1\left(\gamma^5\gamma^\nu C\right)_{\alpha_1\alpha_2}\zeta_{5\nu}\right] = 0 \tag{58}$$

multiplying on the right side by $\left(C^{-1}\gamma^5\right)_{\alpha_2\alpha}$ and considering $\gamma_5^2 = 1$, we have

$$p_\mu\left[a\left(\gamma^\mu\right)_{\alpha\alpha}\zeta_5 - a_1\left(\gamma^\mu\gamma^\nu\right)_{\alpha\alpha}\zeta_{5\nu}\right] = 0 \tag{59}$$

with the use of the trace properties the equation (59) results in

$$a_1\left(p^\mu\zeta_{5\mu}\right) = 0 \tag{60}$$

On the other hand, if we multiply the equation (58) by $\left(C^{-1}\gamma^5\gamma^\lambda\gamma^\rho\right)_{\alpha_2\alpha}$ and taking the trace operation we got to

$$a_1\left(p^\mu\zeta_5^\nu - p^\nu\zeta_5^\mu\right) = 0 \tag{61}$$

for $a_1 \neq 0$, one solution for the last relation is given by

$$\zeta_5^\mu = p^\mu\zeta_5 \tag{62}$$

Thus equations (60) and (61) are the equations for the spin 0 particles and the equation (60) gives the massless Klein-Gordon equation for the scalar field ζ_5.

Now if we multiply (58) on the right side by $\left(C^{-1}\gamma^5\gamma^\lambda\right)_{\alpha_2\alpha}$ we obtain

$$p_\mu\left[a\left(\gamma^\lambda\gamma^\mu\right)_{\alpha\alpha}\zeta_5 - a_1\left(\gamma^\lambda\gamma^\mu\gamma^\nu\right)_{\alpha\alpha}\zeta_{5\nu}\right] = 0 \tag{63}$$

using again the trace properties for the Dirac matrices a third relation is obtained

$$a\left(p^\mu\zeta_5\right) = 0 \tag{64}$$

this equation is compatible with the equation (60) and (61) if only if $a = 0$.

3.2 Spin 1

Now we take the symmetrical part of the vector state $|\Phi\rangle_{\alpha_1\alpha_2}$, the equation (56) becomes

$$(p_\mu \gamma^\mu)_{\alpha\alpha_1} \left[b\,(\gamma^\nu C)_{\alpha_1\alpha_2}\,\zeta_\nu + b_1\left(\Sigma^{\nu\lambda}C\right)_{\alpha_1\alpha_2}\zeta_{\nu\lambda}\right] = 0. \tag{65}$$

Multiplying on the right side by $\left(C^{-1}\gamma^\rho\right)_{\alpha_2\alpha}$ we get

$$p_\mu\left[(\gamma^\mu\gamma^\nu\gamma^\rho)_{\alpha\alpha}\,\zeta_\nu + \left(\gamma^\mu\Sigma^{\nu\lambda}\gamma^\rho\right)_{\alpha\alpha}\zeta_{\nu\lambda}\right] = 0 \tag{66}$$

using the trace properties for the γ^μ-matrices it simplifies to give

$$b_1\left(p^\lambda\zeta_{\lambda\rho}\right) = 0 \tag{67}$$

Multiplying (65) by $\left(C^{-1}\gamma^\rho\gamma^\sigma\gamma^\tau\right)_{\alpha_2\alpha}$ it simplifies to be

$$p_\mu\left[(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau)_{\alpha\alpha}\,\zeta_\nu + \left(\gamma^\mu\Sigma^{\nu\lambda}\gamma^\rho\gamma^\sigma\gamma^\tau\right)_{\alpha\alpha}\zeta_{\nu\lambda}\right] = 0 \tag{68}$$

tracing the equation above and considering the antisymmetric character of the tensor field $\zeta^{\rho\tau}$ we get the Bianchi relation

$$b_1\left(p^\rho\zeta^{\tau\sigma} + p^\sigma\zeta^{\rho\tau} + p^\tau\zeta^{\sigma\rho}\right) = 0 \tag{69}$$

If we set $b_1 \neq 0$, one possible solution of the relation (69) can be obtained if we put

$$\zeta^{\mu\nu} = p^\mu\zeta^\nu - p^\nu\zeta^\mu \tag{70}$$

i.e. the strength tensor of the Maxwell theory and the equation (67) becomes the Maxwell equation for the electromagnetic field .

We can obtain more two equations: the first one is gotten multiplying (65) on the right side by $\left(C^{-1}\right)_{\alpha_2\alpha}$ we have

$$b p_\mu\left(\gamma^\mu\gamma^\nu\right)_{\alpha\alpha}\zeta_\nu = 0 \tag{71}$$

with the help of the trace properties for the Dirac matrices we obtain

$$b\left(p_\mu\zeta^\mu\right) = 0, \tag{72}$$

to get the second one we multiply (65) by $\left(C^{-1}\gamma^\rho\gamma^\sigma\right)_{\alpha_2\alpha}$ and next we take the trace operation over the γ^μ-matrices to obtain

$$b\left(p^\mu\zeta^\nu - p^\nu\zeta^\mu\right) = 0 \tag{73}$$

The equations (72) and (73) are compatible with the equations (67), (69) and (70) if and only if we set $b = 0$.

3.3 Topological solutions

On the other hand we can get two additional solutions if we set $b \neq 0$ and $b_1 = 0$. Thus the first solution is getting when we solve the equation (72) choosing

$$\zeta^\mu = p_\nu\zeta^{\mu\nu} \tag{74}$$

where $\zeta^{\mu\nu}$ is an antisymmetrical tensor field satisfying the equation (73).

And the second solution is founded when set the vector field in the equation (72) being

$$\zeta^\mu = \epsilon^{\mu\nu\alpha\beta}p_\nu\zeta_{\alpha\beta} \tag{75}$$

The equations (74) and (75) are topological field solutions for the spin 1 and spin 0 sectors [34], respectively. Such topological solutions were found in the massless DKP theory by Harish-Chandra [35] and in the context of usual Klein-Gordon and Maxwell theories studying their higher tensor representations by Deser and Witten [36] and Townsend [37].

4 Superspace Formulation

As a natural way we extend the previous analysis of the action and give the formulation in terms of superspace.

Firstly we consider the motion of the particle in the large superspace (big SUSY) (X_μ, Θ_α)[1] whose trajectory is parametrized by the proper supertime (τ, η_1, η_2) of dimension (1/2), here η_1, η_2 are the grassmann real superpartners of the convencional time τ. In this way the coordinates of the particle are scalar superfields in the little superspace (little SUSY). For this case we have[2]

$$\begin{aligned} X_\mu(\tau, \eta_1, \eta_2) &= x_\mu(\tau) + i\eta_i\psi_\mu^i(\tau) + i\eta_i\eta_j F_\mu^{ij}(\tau) &(76)\\ \Theta_\alpha(\tau, \eta_1, \eta_2) &= \theta_\alpha(\tau) + \eta_i\lambda_\alpha^i(\tau) + \eta_i\eta_j\mathcal{F}_\alpha^{ij}(\tau) &(77)\end{aligned}$$

where $i, j = 1, 2$; ψ_μ^i is the grassman superpartner of the common coordinate x_μ; λ_α^i is a commuting majorana spinor, superpartner of the grassmann variables θ_α. $F_\mu^{ij} = -F_\mu^{ji}$ and $\mathcal{F}_\alpha^{ij} = -\mathcal{F}_\alpha^{ji}$ are antisymmetric fields.

In order to construct an action which is invariant under general transformations in superspace we introduce the supereinbein $E_M^A(\tau, \eta_1, \eta_2)$, where M $[A]$ are a curved [tangent] indices and $D_A = E_A^M\partial_M$ is the supercovariant general derivatives, here E_A^M is the inverse of E_M^A. If we take a special gauge

$$E_M^\alpha = \Lambda\overline{E}_M^\alpha, \quad E_M^a = \Lambda^{1/2}\overline{E}_M^a \tag{78}$$

where

$$\overline{E}_\mu^\alpha = 1, \quad \overline{E}_\mu^a = 0, \quad \overline{E}_m^\alpha = -i\eta, \quad \overline{E}_m^a = 1 \tag{79}$$

is the flat space supereinbein, then the superscalar field Λ an the derivative D_A takes the form

$$\begin{aligned} \Lambda(\tau, \eta_1, \eta_2) &= e(\tau) + i\eta_i\chi_i(\tau) + i\eta_i\eta_j f_{ij}(\tau), &(80)\\ \overline{D}_a &\equiv D_i = \frac{\partial}{\partial\eta^i} + i\eta_i\frac{\partial}{\partial\tau}, \quad \overline{D}_\alpha = \partial_\tau &(81)\end{aligned}$$

here $e(\tau)$ is the graviton field and $\chi_i(\tau)$ the gravitino field of the two-dimensional $n = 2$ supergravity; $f_{ij} = -f_{ji}$ is an antisymmetric matrix field. It is no difficult to prove that $\left(\overline{D}_a\right)^2 \equiv (D_i)^2 = i\partial_\tau$

In this way the extension to superspace of the action (6), is given by[3]

$$S = \frac{1}{4}\int d\tau d\eta_1 d\eta_2 \Lambda^{-1}\epsilon_{ij}D_i X_\mu D_j X^\mu \tag{82}$$

here ϵ_{ij} is the antisymmetric matrix: $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. Using the property $\Lambda\Lambda^{-1} = 1$ for the supereinbein field we obtain

$$\begin{aligned} \Lambda^{-1}(\tau, \eta_1, \eta_2) &= e^{-1}(\tau) - ie^{-2}(\tau)\eta_i\chi_i(\tau) - ie^{-2}(\tau)\eta_i\eta_j f_{ij}(\tau)\\ &\quad + e^{-3}(\tau)\eta_i\eta_j\chi_i(\tau)\chi_j(\tau) &(83)\end{aligned}$$

After some manipulations and integrating over the grassmann variables we have

$$\begin{aligned} S &= \int d\tau\left(-\frac{1}{2}e^{-1}\dot{x}^2 + \frac{i}{2}e^{-1}\psi_i\dot{\psi}_i + \frac{i}{2}e^{-2}\chi_i\psi_i\dot{x} + \frac{i}{2}e^{-2}f_{ij}\psi_i\psi_j\right.\\ &\quad \left. + \frac{1}{2}e^{-3}\chi_i\psi_i\chi_j\psi_j + e^{-1}F^2 - ie^{-2}F_{ij}\chi_i\psi_j\right) &(84)\end{aligned}$$

redefining the fields

$$\chi = e^{1/2}\chi', \quad \psi = e^{1/2}\psi', \quad f = ef', \quad F = eF' \tag{85}$$

we obtain

$$\begin{aligned} S &= \int d\tau\left(-\frac{1}{2}e^{-1}\dot{x}^2 + \frac{i}{2}\psi_i\dot{\psi}_i + \frac{i}{2}e^{-1}\chi_i\psi_i\dot{x} + \frac{i}{2}f_{ij}\psi_i\psi_j\right.\\ &\quad \left. + \frac{1}{2}e^{-1}\chi_i\psi_i\chi_j\psi_j + eF^2 - iF_{ij}\chi_i\psi_j\right) &(86)\end{aligned}$$

[1] When the interaction is switched on, we must to include a complex grassmann spinor field $\overline{\Theta}_{\dot{\alpha}}$. This enable us to consider theories with interacting charged particles.

[2] We recall that this form is valid only for the case of two indices $i = 1, 2$. If we want to analyse theories with a bigger internal symmetry $O(N)$, we need to include a more terms.

[3] The presence of the superscalar field Λ is to guarantee the local SUSY invariance.

we see that this action is identical to the proposed in (6) when we put $F = \chi\psi$, i.e. when the fermion coordinate and the gravitino field are coupled.

This shows that considering the correct inclusion of internal symmetries in the superspace formulation we obtain, in the special case, the same action proposed from the pseudoclassical point of view. The internal symmetry group $O(N)$ is connected to the number of grassmann variables η_i.

5 Conclusions

In this work we give an action for the massless DKP theory by using Grassmann variables and the consistence of the equations of motions are assured by means of the inclusion of boundary terms. We also verified the invariance under τ-reparametrizations, local SUSY and internal group $O(N)$ transformations, the generators of these transformations are also found. We carried out the constraint analysis of the theory and verified that after quantization a possible inconsistency can appear, nevertheless the further analysis allow us to solve it with the introduction of some parameters that play a role of regulators of the theory. By the way an important result in this context was obtained, i.e. an additional topological solution for the spin 0 and 1 is derived from this model. As a natural continuation of the presented action we extended the studies to superspace formalism obtaining under some conditions the same initial pseudoclassical action.

For the further development of the theory we are working to accomplish the analysis through the most powerful method for a theory with constraints, i.e. via the BFV-BRST method, which can open the possibility of calculating the propagator of the resulting theory using the path integral representation. And, for further studies the inclusion of interactions (i.e., electromagnetic, Yang-Mills and gravitational fields) in the theory will be discussed.

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